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## ► To cite this version:

Marc Bonnet, Rémi Cornaggia. Higher-order expansion of misfit functional for defect identification in elastic solids. WAVES 2015 The 12th International Conference on Mathematical and Numerical Aspects of Wave Propagation, Jul 2015, Karlsruhe, Germany. hal-01388731

**HAL Id: hal-01388731**

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Submitted on 27 Oct 2016

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## Higher-order expansion of misfit functional for defect identification in elastic solids

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**Abstract.** In this work, least-squares functionals commonly used for defect identification are expanded in powers of the small radius of a trial inclusion, in the context of time-harmonic elastodynamics, generalizing to higher orders the concept of topological derivative. Such expansion, whose derivation and evaluation are facilitated by using an adjoint state, provides a basis for the quantitative estimation of flaws whereby a region of interest may be exhaustively probed at reasonable computational cost.

**Keywords:** Topological derivative, identification, elastodynamics, asymptotic analysis

**Problem statement.** We consider a reference (i.e defect-free) 3D elastic solid  $\Omega$  characterized by Hooke's tensor  $\mathcal{C}$  and mass density  $\rho$ . The time-harmonic background displacement  $\mathbf{u}$  then solves

$$\langle \mathbf{u}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} - \omega^2 \langle \mathbf{u}, \mathbf{w} \rangle_{\Omega}^{\rho} = \mathcal{F}(\mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{W}, \quad (1)$$

where  $\langle \cdot, \cdot \rangle_D^{\mathcal{C}}$  and  $\langle \cdot, \cdot \rangle_D^{\rho}$  denote the stiffness and mass bilinear forms associated to a given domain  $D$  characterized by  $(\mathcal{C}, \rho)$ ,  $\mathcal{W} \subset H^1(\Omega)$  is the function space incorporating the relevant essential boundary conditions (if any), the linear form  $\mathcal{F} \in \mathcal{W}'$  defines the applied time-harmonic loading and  $\omega$  is the angular frequency.

Assuming the presence of a defect inside  $\Omega$ , and that we can measure the resulting displacement  $\mathbf{u}^{\text{ex}}$  on a surface  $\Gamma$ , we define the least-squares cost functional  $J(\mathbf{w})$ , with the elastodynamic displacement  $\mathbf{w}$  associated to the given excitation and a known trial defect, by:

$$J(\mathbf{w}) = \frac{1}{2} \int_{\Gamma} |\mathbf{w}(\mathbf{x}) - \mathbf{u}^{\text{ex}}(\mathbf{x})|^2 dS_{\mathbf{x}} \quad (2)$$

We now consider a specific trial defect  $B_a = \mathbf{z} + a\mathcal{B}$  (Fig 1), centered at  $\mathbf{z} \in \Omega$ , of small size  $a$  and reference shape  $\mathcal{B}$ . It is a perfectly bonded inclusion filled with a material characterized by its Hooke tensor  $\mathcal{C}^* = \mathcal{C} + \Delta\mathcal{C}$  and mass density  $\rho^* = \rho + \Delta\rho$ . We denote  $\mathbf{u}_a$  the displacement in the perturbed domain, and  $\mathbf{v}_a = \mathbf{u}_a - \mathbf{u}$  the displacement perturbation.  $J(\mathbf{u}_a)$  admits the

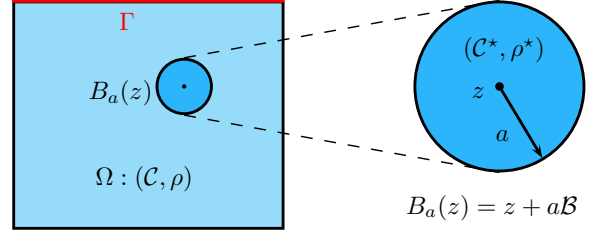


Figure 1: Computational domain and inclusion.

exact expansion about  $\mathbf{u}$ :

$$\begin{aligned} J(\mathbf{u}_a) &= J(\mathbf{u}) + J'(\mathbf{u}; \mathbf{v}_a) + J''(\mathbf{u}; \mathbf{v}_a, \mathbf{v}_a) \\ &= J(\mathbf{u}) + \Re \int_{\Gamma} \overline{(\mathbf{u} - \mathbf{u}^{\text{ex}})} \cdot \mathbf{v}_a + \frac{1}{2} \int_{\Gamma} |\mathbf{v}_a|^2 \end{aligned} \quad (3)$$

The goal is now to expand  $J(\mathbf{u}_a)$  in powers of  $a$ . Similar expansions have been studied in e.g. [2] for rigid obstacles in 3D acoustic media and [5] for holes in 2D elastic bodies.

Define the adjoint field  $\mathbf{p}$  as the solution of

$$\langle \mathbf{p}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} - \omega^2 \langle \mathbf{p}, \mathbf{w} \rangle_{\Omega}^{\rho} = J'(\mathbf{u}; \mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{W}. \quad (4)$$

We can then compute  $J'(\mathbf{u}; \mathbf{v}_a)$  as

$$J'(\mathbf{u}; \mathbf{v}_a) = -\langle \mathbf{p}, \mathbf{u}_a \rangle_{B_a}^{\Delta\mathcal{C}} + \omega^2 \langle \mathbf{p}, \mathbf{u}_a \rangle_{B_a}^{\Delta\rho} \quad (5)$$

Expanding  $\mathbf{v}_a$  in powers of  $a$  is now needed. As we will see,  $J''(\mathbf{u}; \mathbf{v}_a, \mathbf{v}_a)$  requires only the leading contribution of  $\mathbf{v}_a|_{\Gamma}$  whereas a higher-order expansion of  $\mathbf{v}_a|_{B_a}$  is needed for evaluating (5).

### Expansion of the solution perturbation.

Following e.g. [3],  $\mathbf{v}_a$  solves the integro-differential Lippmann-Schwinger equation:

$$\mathcal{L}_a[\mathbf{v}_a](\mathbf{x}) = -\langle \mathbf{u}, \mathbf{G} \rangle_{B_a}^{\Delta\mathcal{C}} + \omega^2 \langle \mathbf{u}, \mathbf{G} \rangle_{B_a}^{\Delta\rho} \quad (6)$$

with  $\mathcal{L}_a[\mathbf{v}](\mathbf{x}) := \mathbf{v}(\mathbf{x}) + \langle \mathbf{v}, \mathbf{G} \rangle_{B_a}^{\Delta\mathcal{C}} - \omega^2 \langle \mathbf{v}, \mathbf{G} \rangle_{B_a}^{\Delta\rho}$  and  $\mathbf{G} = \mathbf{G}(\cdot, \mathbf{x})$  is the elastodynamic Green's tensor for a unit point force applied at  $\mathbf{x}$  and satisfying homogeneous boundary conditions consistent with problem (1) on  $\partial\Omega$ . Substituting the ansatz

$$\begin{aligned} \mathbf{v}_a(\mathbf{x}) &= a\mathbf{V}_1(\bar{\mathbf{x}}) + a^2\mathbf{V}_2(\bar{\mathbf{x}}) + \frac{1}{2}a^3\mathbf{V}_3(\bar{\mathbf{x}}) \\ &\quad + \frac{1}{6}a^4\mathbf{V}_4(\bar{\mathbf{x}}) + \delta_a(\mathbf{x}), \quad \mathbf{x} \in B_a \end{aligned} \quad (7)$$

(with  $\bar{x} := (\mathbf{x} - \mathbf{z})/a \in \mathcal{B}$ ) into (6) and expanding the resulting equation in powers of  $a$  (in particular using that  $\mathbf{G}(\boldsymbol{\xi}, \mathbf{x}) = a^{-1}\mathbf{G}_\infty(\bar{\boldsymbol{\xi}} - \bar{\mathbf{x}}) + O(1)$ , with  $\mathbf{G}_\infty$  denoting the *static* full-space Kelvin fundamental solution) yields a sequence of integral equations for the  $\mathbf{V}_j$ . These equations correspond to *elastostatic* problems for the normalized inclusion  $\mathcal{B}$  embedded in an unbounded reference medium, and are solved with the help of Eshelby's equivalent inclusion method [4].

The remainder  $\boldsymbol{\delta}_a$  in (7) solves an integro-differential equation of the form  $\mathcal{L}_a[\boldsymbol{\delta}_a] = \boldsymbol{\gamma}_a$ . The operator  $\mathcal{L}_a : H^1(B_a) \rightarrow H^1(B_a)$  is shown to be invertible with bounded inverse, while  $\boldsymbol{\gamma}_a$  can be estimated as  $\|\boldsymbol{\gamma}_a\|_{H^1(B_a)} = O(a^{11/2})$ . Consequently, there exists a constant  $C > 0$  independent of  $a$  such that

$$\|\boldsymbol{\delta}_a\|_{H^1(B_a)} \leq C a^{11/2}. \quad (8)$$

For  $\mathbf{x} \notin B_a$ , plugging (7) in the form  $\mathbf{v}_a(\mathbf{x}) \approx a\mathbf{V}_1(\bar{\mathbf{x}})$  into (6) yields the outer expansion

$$\begin{aligned} \mathbf{v}_a(\mathbf{x}) = & -a^3 [\boldsymbol{\nabla} \mathbf{u}(\mathbf{z}) : \mathcal{A} : \boldsymbol{\nabla} \mathbf{G}(\mathbf{z}, \mathbf{x}) \\ & - \omega^2 \Delta \rho |\mathcal{B}| \mathbf{u}(\mathbf{z}) \cdot \mathbf{G}(\mathbf{z}, \mathbf{x})] + o(a^3), \end{aligned} \quad (9)$$

$\mathcal{A}$  being the elastic moment tensor associated to  $\mathcal{B}$ ,  $\mathcal{C}$  and  $\Delta \mathcal{C}$  [1, 3].

**Cost functional expansion.** Substituting (7) into (5) and (9) into (3),  $J(\mathbf{u}_a)$  is finally found to have an expansion of the form:

$$J(\mathbf{u}_a) = J_6(a, \mathbf{z}) + o(a^6) \quad (10)$$

$$\begin{aligned} \text{with } J_6(a, \mathbf{z}) = & J(\mathbf{u}) + a^3 \mathcal{T}_3(\mathbf{z}) + a^4 \mathcal{T}_4(\mathbf{z}) \\ & + a^5 \mathcal{T}_5(\mathbf{z}) + a^6 \mathcal{T}_6(\mathbf{z}), \end{aligned}$$

the  $o(a^6)$  estimate resulting from (8) and (9).

The  $\mathcal{T}_j(\mathbf{z})$  are found to be given in terms of (i) the background field  $\mathbf{u}$  and its derivatives at  $\mathbf{z}$ , (ii) the adjoint field  $\mathbf{p}$  and its derivatives at  $\mathbf{z}$ , (iii)  $\mathcal{A}$  and other elastic moment tensors that involve the material parameters, the shape  $\mathcal{B}$  and the angular frequency  $\omega$ , and (iv) the complementary part of  $\mathbf{G}$ , i.e.  $\mathbf{G} - \mathbf{G}_\infty = (\mathbf{G}_{\infty, \omega} - \mathbf{G}_\infty) + \mathbf{G}_C$ , where  $\mathbf{G}_{\infty, \omega}$  is the elastodynamic full-space fundamental solution and  $\mathbf{G}_C$  accounts for the boundedness of  $\Omega$ . In particular,  $\mathcal{T}_3(\mathbf{z})$  is the well-known topological derivative:

$$\mathcal{T}_3(\mathbf{z}) = -[\boldsymbol{\nabla} \mathbf{u} : \mathcal{A} : \boldsymbol{\nabla} \mathbf{p} - \omega^2 \Delta \rho |\mathcal{B}| \mathbf{u} \cdot \mathbf{p}](\mathbf{z}).$$

Moreover,  $\mathcal{T}_4(\mathbf{z}) = 0$  for any centrally-symmetric shape  $\mathcal{B}$ . The complementary part  $\mathbf{G}_{\infty, \omega} - \mathbf{G}_\infty$  (known analytically) is involved in  $\mathcal{T}_5(\mathbf{z})$  and

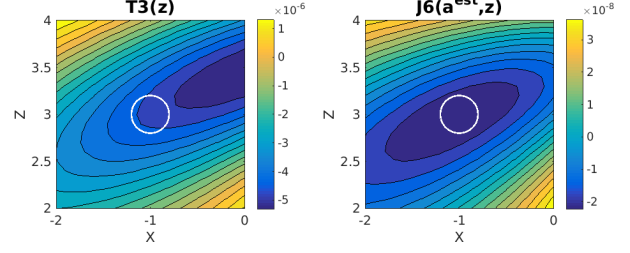


Figure 2:  $\mathcal{T}_3(\mathbf{z})$  and  $J_6(a^{\text{est}}, \mathbf{z})$  plotted in the  $(XZ)$  plane around the obstacle (in white).

$\mathcal{T}_6(\mathbf{z})$ , while  $\mathbf{G}_C$  appears in  $\mathcal{T}_6(\mathbf{z})$  only. Since the exact computation of  $\mathbf{G}_C$  would require solving an elastodynamic problem on  $\Omega$  for each trial location  $\mathbf{z}$ , we plan to use an approximation method to save computational time.

Closed-form formulae for the  $\mathcal{T}_j$  can be obtained when  $\mathcal{B}$  is spherical (for which case we provide explicit expressions) or ellipsoidal.

**Identification.** Following [2], estimates of the location  $\mathbf{z}^{\text{est}}$  and size  $a^{\text{est}}$  of the real defect can then be sought as minimizers of  $J_6(a, \mathbf{z})$ , with  $\mathbf{z}$  spanning a predefined sampling grid. This entails computing the  $\mathcal{T}_j(\mathbf{z})$  over the sampling grid and minimizing  $a \mapsto J_6(a, \mathbf{z})$  for each  $\mathbf{z}$ , the latter step being very fast and straightforward.

A preliminary example is set in free space (so that  $\mathbf{G}_C = 0$ ) for a spherical scatterer of radius  $0.1\lambda_S$  illuminated by a plane P-wave travelling along the positive  $x$ -direction, with a discrete array of displacement sensors lying behind the scatterer. The above procedure yields the size estimate  $a^{\text{est}} \approx 0.105\lambda_S$ ; moreover, the estimated location  $\mathbf{z}^{\text{est}}$  is found to be very close to the true center of the scatterer. The contour plot of  $J_6(a^{\text{est}}, \mathbf{z})$  (Fig 2) shows improved localisation (relative to the topological derivative  $\mathcal{T}_3(\mathbf{z})$ ) for this partial-aperture configuration.

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